

# Degrees of Freedom of a Communication Channel and Kolmogorov numbers

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**Abstract**—In this note, we show that the operator theoretic concept of Kolmogorov numbers and the number of degrees of freedom at level  $\epsilon$  of a communication channel are closely related. Linear communication channels may be modeled using linear compact operators on Banach or Hilbert spaces and the number of degrees of freedom of such channels is defined to be the number of linearly independent signals that may be communicated over this channel, where the channel is restricted by a threshold noise level. Kolmogorov numbers are a particular example of  $s$ -numbers, which are defined over the class of bounded operators between Banach spaces. We demonstrate that these two concepts are closely related, namely that the Kolmogorov numbers correspond to the “jump points” in the function relating numbers of degrees of freedom with the noise level  $\epsilon$ . We also establish a useful numerical computation result for evaluating Kolmogorov numbers of compact operators.

straints such as finite transmission power and noise at the receiver, only finitely many (linearly independent) signals may be exchanged between the transmitter and receiver. This physical intuition may be captured using the concept of number of degrees of freedom of the communication channel. The number of degrees of freedom of a communication channel has been used in evaluating the Shannon capacity for several physically realistic channels<sup>1</sup> (see e.g. [1], [2], [3], [4]). This concept has also been used in various other problem domains such as multi-antenna communication [5], [6], [7], optics [8], [9] and electromagnetic field sampling [10].

Kolmogorov numbers are a particular example of so called  $s$ -number sequences. In the theory of  $s$ -numbers one associates with every bounded linear operator  $T$ , mapping between any two Banach spaces, a scalar sequence  $s_1(T) \geq s_2(T) \geq \dots \geq 0$  (see e.g. [11], [12]). A classical example in the more

## I. INTRODUCTION

The number of degrees of freedom of a communication channel plays an important role in evaluating the channel’s Shannon capacity (see e.g. [1]). In a physical communication system, because of con-

<sup>1</sup>The concept of number of degrees of freedom is used implicitly in evaluating the capacity of Shannon’s classical bandwidth limited, additive Gaussian white noise channel. In particular it can be shown that as the bandwidth becomes large the number of degrees of freedom at level  $\epsilon$  approaches the well know constant  $2WT$ , for all noise levels  $\epsilon$  (see Gallager [4, Ch. 8]).

restricted category of compact operators mapping between Hilbert spaces is the sequence of singular values.  $s$ -number sequences of various types have primarily been used to classify operators based on the behaviour of the sequence  $s_n$  as  $n \rightarrow \infty$ . In particular, various interesting operator ideals have been obtained based on the behaviour of these sequences.

In this note we establish the connection between Kolmogorov numbers and degrees of freedom of a communication channel. The remainder of this note is organised as follows: in Section II, we recall the definitions of degrees of freedom of a communication channel and briefly explain the physical intuition behind the definition. We also recall the definitions of  $s$ -number sequences in general and Kolmogorov numbers in particular. We present our main results, establishing the connection between the number of degrees of freedom and Kolmogorov numbers, in Section III and provide conclusions in the final Section IV.

## II. DEFINITIONS

### A. Degrees of Freedom

The number of degrees of freedom of a communication channel has for example been defined in [1]. The physical model used in [1] is as follows: there is a normed vector space  $X$  of transmitter functions and a normed vector space  $Y$  of receiver functions and a channel operator  $T : X \rightarrow Y$  that is a compact linear operator. Physically, the elements of  $X$  can be thought of as signals that a transmitter generates, and  $T$  maps these signals onto elements of  $Y$ , which is the space of signals a receiver can measure. It is further assumed that there are two constraints on this channel: 1) the power available for transmission

is finite. This constraint is modeled by assuming that the transmitter may only generate elements of  $X$  that have a norm less than or equal to 1. 2) the receiver is noisy and therefore small signals can not be measured. This constraint is modeled by assuming that only received signals with norm greater than some pre-specified small constant  $\epsilon$  may be measured.

It was shown in [1] that the following definition may be used for the number of degrees of freedom of a compact operator that models a communication channel.

*Definition 2.1 (Degrees of freedom at level  $\epsilon$ ):*

Suppose  $X$  and  $Y$  are normed spaces with norms  $\|\cdot\|_X$  and  $\|\cdot\|_Y$ , respectively, and  $T : X \rightarrow Y$  is a compact operator. Also, let  $\overline{B}_{r,X}(\theta)$  denote the closed ball of radius  $r$  centered at  $\theta$  in  $X$ . Then the number of degrees of freedom  $\mathcal{N}(\epsilon)$  of  $T$  at level  $\epsilon$  is the smallest<sup>2</sup>  $N \in \mathbb{Z}_0^+$  such that there exists a set of vectors  $\{\psi_1, \dots, \psi_N\} \subset Y$  such that for all  $x \in \overline{B}_{1,X}(0)$

$$\inf_{a_1, \dots, a_N} \left\| Tx - \sum_{i=1}^N a_i \psi_i \right\|_Y \leq \epsilon.$$

It was shown in [1] that the number of degrees of freedom is well defined. Physically, we interpret this definition as follows: if there is some constraint  $\|\cdot\|_X \leq 1$  on the space of transmitter functions<sup>3</sup> and if the receiver can only measure signals that satisfy  $\|\cdot\|_Y > \epsilon$ , then the number of degrees of freedom is the maximum number of linearly independent

<sup>2</sup> $\mathbb{Z}_0^+$  denotes the set of non-negative integers.

<sup>3</sup>We normalised the transmitter power constraint by assuming that available transmission power,  $P = 1$ . Different normalisations can e.g. be achieved by re-scaling the norm on  $X$ .

signals that the receiver can distinguish under these constraints, where the maximum is taken over all possible signal constellations.

The following theorem is a simple consequence of the above definition.

**Theorem 2.1:** [1, Th. 3.2] Suppose  $X$  and  $Y$  are normed spaces with norms  $\|\cdot\|_X$  and  $\|\cdot\|_Y$ , respectively, and  $T : X \rightarrow Y$  is a compact operator. Let  $\mathcal{N}(\epsilon)$  denote the number of degrees of freedom of  $T$  at level  $\epsilon$ . Then

- 1)  $\mathcal{N}(\epsilon) = 0$  for all  $\epsilon \geq \|T\|$ .
- 2) Unless  $T$  is identically zero, there exists an  $\epsilon_0 > 0$  such that  $\mathcal{N}(\epsilon) \geq 1$  for all  $0 < \epsilon < \epsilon_0$ .
- 3)  $\mathcal{N}(\epsilon)$  is a non-increasing, upper semicontinuous function of  $\epsilon$ .
- 4) In any finite interval  $(\epsilon_1, \epsilon_2) \subset \mathbb{R}$ , with  $0 < \epsilon_1 < \epsilon_2$ ,  $\mathcal{N}(\epsilon)$  has only finitely many discontinuities, i.e.  $\mathcal{N}(\epsilon)$  only takes finitely many non-negative integer values in any finite  $\epsilon$  interval.

### B. $s$ -numbers and Kolmogorov numbers

The axiomatic characterisation of  $s$ -numbers for Banach space valued operators is due to Pietsch, cf. [12]. In the remainder of this section, we assume that  $X, X', Y$  and  $Y'$  are Banach spaces,  $\mathcal{B}(X, Y)$  denotes the set of bounded linear operators  $T : X \rightarrow Y$  and  $\|\cdot\|$  is the standard operator norm on  $\mathcal{B}(X, Y)$ . Also, we denote by  $I$  the identity operator.

**Definition 2.2:** Let  $s : T \mapsto (s_n(T))_{n=1}^\infty$  be a rule that assigns to every bounded operator  $T$  on some Banach space a scalar sequence such that the following conditions are satisfied:

- SN1  $\|T\| = s_1(T) \geq s_2(T) \geq \dots \geq 0$ .
- SN2  $s_n(S + T) \leq s_n(S) + \|T\|$ , for  $S, T \in \mathcal{B}(X, Y)$  and  $n = 1, 2, \dots$

$$\text{SN3 } s_n(BTA) \leq \|B\|s_n(T)\|A\|, \text{ for } A \in \mathcal{B}(X', X), T \in \mathcal{B}(X, Y), B \in \mathcal{B}(Y, Y') \text{ and } n = 1, 2, \dots$$

$$\text{SN4 } s_n(I) = 1, \text{ for } n = 1, 2, \dots$$

$$\text{SN5 } s_n(T) = 0 \text{ if } \text{rank}(T) < n.$$

Then  $(s_n(T))_{n=1}^\infty$  is called an  $s$ -number sequence.

It turns out that in the compact, Hilbert space case, i.e. if we restrict the above definition to the class of compact operators mapping between Hilbert spaces, the above properties SN1–SN5 uniquely characterise the singular values of the operator (ordered in descending order).

For any linear subspace  $S$  of  $Y$  let  $Q_S^Y$  denote the natural surjection from  $Y$  onto the quotient space  $Y/S$ . The numbers

$$d_n(T) \triangleq \inf\{\|Q_S^Y T\| : \dim(S) < n\}$$

for  $T \in \mathcal{B}(X, Y)$  define an  $s$ -number sequence [12], and are called *Kolmogorov numbers*.

## III. MAIN RESULTS

Let  $X$  and  $Y$  be Banach spaces with norms  $\|\cdot\|_X$  and  $\|\cdot\|_Y$ , respectively, let  $T : X \rightarrow Y$  be compact<sup>4</sup> and suppose  $B(X)$  is the unit ball in  $X$ . Let  $\mathcal{N}(\epsilon)$  denote the number of degrees of freedom of  $T$  at level  $\epsilon$  and let  $d_n = d_n(T)$  denote the  $n^{\text{th}}$  Kolmogorov number.

For ease of notation we set  $\mathcal{N}(\epsilon) = \infty$  for  $\epsilon < 0$  for the remainder of this document. Now, consider

<sup>4</sup>We only consider compact  $T$ , because the number of degrees of freedom is only defined for compact operators. One can conclude from the arguments in [1] that for physically realistic models of communication channels, the channel operator must be compact. However, it may be possible to generalise the theory presented in the current paper to bounded operators that are not necessarily compact.

the sequence  $\{\sigma_n = \sigma_n(T)\}, n = 1, 2, \dots$  implicitly defined as follows

$$\sup_{\epsilon > \sigma_n} \mathcal{N}(\epsilon) = n - 1 \quad \text{and} \\ \inf_{\epsilon < \sigma_n} \mathcal{N}(\epsilon) = N \geq n.$$

Further, if  $n < N$  then for all  $m$  such that  $n < m \leq N$ ,  $\sigma_m \triangleq \sigma_n$ . The sequence  $\{\sigma_n\}$  was defined in [1] and  $\sigma_n$  was called the  $n^{\text{th}}$  DOF singular value of  $T$  in [1]. It should be noted that in the Hilbert space situation,  $\sigma_n$  is simply the  $n^{\text{th}}$  singular value of  $T$ . Also,  $\mathcal{N}(\epsilon)$  is discontinuous exactly at  $\epsilon = \sigma_n$  for  $n = 1, 2, \dots$ . We now show that  $\sigma_n$  is in fact equal to  $d_n$  for which we will need the following elementary lemma.

**Lemma 3.1:**

$$d_{n+1} = \inf_{\psi_1, \dots, \psi_n \in Y} \sup_{x \in B(X)} \inf_{a_1, \dots, a_n \in \mathbb{R}} \left\| Tx - \sum_{i=1}^n a_i \psi_i \right\|$$

**Proof:** Let  $S$  be any subspace of  $Y$  such that  $\dim(S) \leq n$ . By the definition of  $Q_S^Y$ ,

$$\|Q_S^Y T\| = \sup_{x \in B(X)} \inf_{y \in S} \|Tx - y\|_Y.$$

The lemma now follows from two simple facts: for  $\dim(S) \leq n$  there exists a set of vectors  $\{\psi_1, \dots, \psi_n\}$  such that they span  $S$ , and the dimension of the span of  $n$  vectors is not greater than  $n$ . ■

**Theorem 3.2:**  $\sigma_n = d_n$ .

**Proof:** Because  $\sigma_1 = d_1 = \|T\|$ , we only need to prove that  $\sigma_{n+1} = d_{n+1}$  for  $n = 1, 2, \dots$ .

Now suppose  $\sigma_N = \sigma_{N-1} = \dots = \sigma_{n+1} < \sigma_n$ . We first prove that  $\sigma_{n+1} \geq d_{n+1}$ . Let  $\delta > 0$  be some arbitrary number satisfying  $\delta < \sigma_n - \sigma_{n+1}$ . Then for  $\epsilon = \sigma_{n+1} + \delta$ ,

$$\mathcal{N}(\epsilon) = n.$$

Therefore, there exists a set  $\{\psi_1, \dots, \psi_n\} \in Y$  such that

$$\sup_{x \in B(X)} \inf_{a_1, \dots, a_n} \left\| Tx - \sum_{i=1}^n a_i \psi_i \right\|_Y \leq \epsilon = \sigma_{n+1} + \delta.$$

Because, the constant  $\delta > 0$  can be made arbitrarily small,

$$d_{n+1} = \inf_{\psi_1, \dots, \psi_n \in Y} \sup_{x \in B(X)} \inf_{a_1, \dots, a_n \in \mathbb{R}} \left\| Tx - \sum_{i=1}^n a_i \psi_i \right\| \leq \sigma_{n+1}. \quad (1)$$

Relabeling  $N = n + 1$ , we get

$$d_N \leq \sigma_N = \sigma_{n+1}.$$

To prove the converse inequality, let  $\epsilon < \sigma_N$ . Then,  $\mathcal{N}(\epsilon) > N$ . Therefore, for all sets  $\{\psi_1, \dots, \psi_N\} \subset Y$ , there exists an  $x \in B(X)$  such that,

$$\inf_{a_1, \dots, a_N} \left\| Tx - \sum_{i=1}^N a_i \psi_i \right\|_Y > \epsilon > \sigma_N = \sigma_{n+1}.$$

Therefore,

$$\begin{aligned} d_{n+1} &\geq d_N \\ &= \inf_{\psi_1, \dots, \psi_N \in Y} \sup_{x \in B(X)} \inf_{a_1, \dots, a_N \in \mathbb{R}} \left\| Tx - \sum_{i=1}^N a_i \psi_i \right\| \\ &\geq \sigma_N \\ &= \sigma_{n+1}. \end{aligned} \quad (2)$$

Here, we used the  $s$ -number property SN1 that  $d_n$  is non-increasing in  $n$ . The result now follows from (1) and (2). ■

**Remark 3.1:** The number of degrees of freedom of a communication channel is defined for compact operators mapping between arbitrary normed spaces, while the concept of Kolmogorov numbers is defined only for operators mapping between Banach spaces. However, if  $X$  and  $Y$  are normed spaces and  $T : X \rightarrow Y$  is compact, then consider the completions  $\tilde{X}$  and  $\tilde{Y}$  of  $X$  and  $Y$ , respectively, so that  $\tilde{X}$  and  $\tilde{Y}$  are Banach spaces. Let  $\tilde{T} : \tilde{X} \rightarrow \tilde{Y}$  be the standard extension of  $T$  from  $X$  to  $\tilde{X}$ . Then  $\sigma_n(T) = \sigma_n(\tilde{T})$ . This fact follows directly from the compactness of  $T$  and the definition of  $\mathcal{N}(\cdot)$ .

We also restate below a useful result in [1] that aids in the numerical computation of Kolmogorov numbers for compact operators.

**Theorem 3.3:** [1, Th. 3.8] Suppose  $X$  and  $Y$  are Banach spaces and  $T : X \rightarrow Y$  is a compact operator. Also suppose that  $X$  has a complete Schauder basis  $\{\phi_1, \phi_2, \dots\}$  and let  $S_m = \text{span}\{\phi_1, \dots, \phi_m\}$ . Let  $T_m = T|_{S_m} : S_m \rightarrow Y$ ,  $m \in \mathbb{Z}^+$ . If  $\sigma_n$ , the  $n^{\text{th}}$  DOF singular value of  $T$ , exists then for  $m$  large enough  $\sigma_{n,m}$ , the  $n^{\text{th}}$  DOF singular value of  $T_m$ , will exist and

$$\lim_{m \rightarrow \infty} \sigma_{n,m} = \sigma_n.$$

If  $\sigma_{n,m}$  exists then it is a lower bound for  $\sigma_n$ .

We can therefore use finite dimensional approximations of  $T$  to numerically compute the Kolmogorov numbers.

#### IV. CONCLUSION

In this note we establish the connection between Kolmogorov numbers and degrees of freedom of a communication channel. Specifically, we show that the jump points (discontinuities) of the function relating the number of degrees of freedom to the noise level  $\epsilon$  are equal to the Kolmogorov numbers. This connection invites the question as to whether other  $s$ -number sequences such as Gelfand numbers or approximation numbers play an equally important role in communication systems.

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